Power operations in K-theory completed at a prime

Andrew Baker *

School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, Scotland
E-mail: a.baker@maths.gla.ac.uk

Abstract

We describe the action of power operations on the p-completed cooperation algebras $K_0^\vee K = K_0(K)\widehat{_p}$ for K-theory at a prime p, and $K_0^\vee KO = K_0(KO)\widehat{_2}$. These results are used to identify the $K(1)$-local homotopy type of some E_∞ ring spectra obtained by killing elements of Hopf invariant 1.

2000 Mathematics Subject Classification. **55P43**. 13D03, 55N35, 55P48.
Keywords. K-theory, E_∞ ring spectrum, commutative S-algebra, power operation.

Introduction

Power operations in suitably completed (co)homology theories have been studied and used by several authors, for example Rezk [29, 30, 31]; the paper of Barthel and Frankland [11] building on work of McClure [13] provides a convenient account of this, in particular for the case of p-complete K-theory. An important source on related mathematics is the article by Hopkins [18], and indeed the volume [15] contains much that the reader may find helpful.

In the present paper we describe the action of the θ-operator (which we follow [11] in denoting by Q) on the p-completed cooperation algebra

$$K_0^\vee K = K_0(K)\widehat{_p} = \pi_0(L_{K(1)}(K \wedge K)),$$

where $K = KU$. We expect this to be of use in investigating the θ-action and its interaction with the $K_*^\vee(K)$-coaction on $K_*^\vee(A)$ for any $\mathcal{E}_\infty$ ring spectrum A. We also give some results on $K_0^\vee(KO)$ when $p = 2$ and on $K_*^\vee(\mathbb{P}X)$, where $\mathbb{P}X$ denotes the free commutative S-algebra on a spectrum X introduced in [16].

It is likely that some of our results are known to experts, but we have not found a published source, so we feel it worthwhile writing them down.

An obvious related problem to investigate is that of describing the actions of power operations on $K_0^\vee(BU)$ or equivalently on $K_0^\vee(MU)$ (these actions correspond under the Thom isomorphism). The $\mathcal{E}_\infty$ orientation of [22] induces a morphism of θ-algebras $K_0^\vee(MU) \to K_0^\vee(K)$ but this is not injective on the image of the Hopf algebra primitives $\Pr K_0^\vee(BU)$, and this seems to make the determination of the action on primitives more delicate then in the case of ordinary mod p homology as carried out by Kochman [23]. We may return to this in future work.

*The mathematics described in this paper is based in part on work supported by the National Science Foundation under Grant No. 0932078 000 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley California, during the Spring 2014 semester; the author also acknowledges the support of the Max Planck Institute for Mathematics in Bonn during a visit in January 2020.
I would like to thank Gerd Laures, Justin Noel and Charles Rezk for helpful conversations, and especially Francis Clarke who taught me about the important rôle of p-adic analysis in K-theory back in the early 1980s.

Tbilisi Mathematical Journal Special Issue (HomotopyTheorySpectra - 2020), pp. 33–49.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 28 January 2020.
Accepted for publication: 30 July 2020.

Conventions and notation: We will work with $\mathcal{E}_\infty$ ring spectra in the setting of commutative S-algebras of [16] and use these terms interchangeably. We will assume that KU and KO have their standard $\mathcal{E}_\infty$ ring structures as produced in [10] for example.

Throughout, p will be a fixed prime and $K = KU_{(p)}$ will denote the p-local 2-periodic complex K-theory ring spectrum; we will also denote the p-adic completion of K by $K_p^\wedge = KU_p^\wedge$. We will often denote (co)homology without brackets where appropriate by setting $K^*X = K^*(X)$ and $K_*X = K_*(X)$ for example, but include brackets where it improves readability.

1 L-complete modules

We will be working with p-complete K-theory for a prime p, and this takes values in the category of L-complete graded modules for the local ring $\mathbb{Z}_{(p)}$. The utility of working with such a category originated in work of Greenlees & May [17] and was made explicit by Hovey & Strickland [21]. The reader is also referred to Barthel & Frankland [11] for a more recent account.

A fundamental observations is that for any spectrum each p-completed K-theory group

$$K_n^\vee X = \pi_n(L_{K(1)}(K \wedge X))$$

is L-complete (with respect to $\mathbb{Z}_{(p)}$), i.e., $K_n^\vee X \cong L_0 K_n^\vee X$ where L_s ($s \geqslant 0$) is the left derived functor of p-adic completion on the category of $\mathbb{Z}_{(p)}$-modules. In fact L_s is trivial when $s > 1$.

When M is $\mathbb{Z}_{(p)}$-free or flat then $L_0 M = M_p^\wedge$ and $L_1 M = 0$ by [6]. More generally, $L_* M$ can calculated by taking a free resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow 0$$

and taking homology of the induced complex

$$0 \leftarrow (F_0)_p^\wedge \leftarrow (F_1)_p^\wedge \leftarrow 0.$$

For $M = K_n(X)$ this allows us to induce up the effect of a natural transformation $\theta\colon K_n(-) \to K_n(-)$. To see how to do this we need some background.

Recall that a ring spectrum E satisfies the *Adams condition* of [1] if it can be written as colimit $E = \operatorname{colim}_\alpha E_\alpha$ of dualisable spectra E_α. This condition ensures the existence of suitable resolutions for constructing Universal Coefficient spectral sequences.

In particular, KU and KO satisfy the Adams condition, see [1, proposition 13.4]. The proof there uses even suspensions of skeleta of BU and BSp (with cells in even degrees); in fact these can be replaced by suspensions of skeleta of $\mathbb{C}\mathrm{P}^\infty$ and $\mathbb{H}\mathrm{P}^\infty$ by results of [3].

Then the K_*-module $K_*(X)$ can be resolved using the following procedure due to Adams, see [1, lemma 13.7]. Take a set of K_*-module generators of $K_*(X) = \operatorname{colim}_\alpha \pi_*(K_\alpha \wedge X)$ and form their adjoint maps $f\colon \Sigma^{n(f)} DE_\alpha \to X$ so that together these induce an epimorphism

$$\bigoplus_f K_*(DE_\alpha) = K_*\Big(\bigvee_f DE_\alpha\Big) \xrightarrow{\varepsilon} K_* X.$$

Here each $K_*(DE_\alpha)$ is a finitely generated free K_*-module and by work of Hovey [19, theorem 3.3],

$$K_*^\vee\Big(\bigvee_f DE_\alpha\Big) \cong \Big(\bigoplus_f K_*(DE_\alpha)\Big)_p^\wedge.$$

which is pro-free. As K_* is a graded principal ideal domain, $\ker\varepsilon$ is also a free K_*-module, so $L_*K_*(X)$ can be calculated using the complex

$$0 \leftarrow K_*^\vee\Big(\bigvee_f DE_\alpha\Big) \leftarrow (\ker\varepsilon)\hat{}_p \leftarrow 0.$$

Notice also that the spectral sequence of [20, corollary 3.2] collapses to give a collection of short exact sequences

$$0 \to L_0K_n(X) \to K_*^\vee(X) \to L_1K_{n-1}(X) \to 0.$$

2 K-theory completed at a prime and power operations

We first recall some standard facts about the rings of p-local integers $\mathbb{Z}_{(p)}$ and p-adic integers $\mathbb{Z}_p$. By definition, if we give $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$ the p-adic norm topologies then $\mathbb{Z}_{(p)} \subseteq \mathbb{Z}_p$ is a dense subring. The residue fields of $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$ both agree with the finite field $\mathbb{F}_p$ which we give the discrete topology. There is a pullback square of topological multiplicative monoids

$$\begin{tikzcd}
\mathbb{Z}_{(p)}^\times \arrow[r, hook] \arrow[d, two heads] & \mathbb{Z}_{(p)} \arrow[d, two heads] \\
\mathbb{F}_p^\times \arrow[r, hook] & \mathbb{F}_p
\end{tikzcd}$$

and on p-adic completion this becomes the pullback square

$$\begin{tikzcd}
\mathbb{Z}_p^\times \arrow[r, hook] \arrow[d, two heads] & \mathbb{Z}_p \arrow[d, two heads] \\
\mathbb{F}_p^\times \arrow[r, hook] & \mathbb{F}_p
\end{tikzcd}$$

so $\mathbb{Z}_p^\times$ is the completion of $\mathbb{Z}_{(p)}^\times$ with respect to the p-adic norm.

It is known from [3, 2, 4, 5] that

$$K_0K \cong \{f(w) \in \mathbb{Q}[w, w^{-1}] : f(\mathbb{Z}_{(p)}^\times) \subseteq \mathbb{Z}_{(p)}\},$$

and K_0K is a free $\mathbb{Z}_{(p)}$-module. Since $\mathbb{Z}_{(p)}^\times$ is a dense subgroup of $\mathbb{Z}_p^\times$, we may interpret Laurent polynomials as continuous functions on $\mathbb{Z}_p^\times$ and obtain

$$K_0K \cong \{f(w) \in \mathbb{Q}[w, w^{-1}] : f(\mathbb{Z}_p^\times) \subseteq \mathbb{Z}_p\} \subseteq \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p),$$

where the latter is the p-adic Banach algebra of continuous maps $\mathbb{Z}_p^\times \to \mathbb{Z}_p$ equipped with the operator norm; it is known that this subring of $\mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ is dense. It follows that

$$K_0^\vee K = \pi_0((K \wedge K)\hat{}_p) = (K_0K)\hat{}_p,$$

where the p-adic topology involved in the completion agrees with p-adic norm topology inherited from $\mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$. Therefore there is an isomorphism of p-adic Banach algebras

$$K_0^\vee K \cong \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p). \tag{2.1}$$

For $a \in \mathbb{Z}_{(p)}^{\times}$, the stable Adams operation

$$\psi^a \in K^0K \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}}(K_0K, \mathbb{Z}_{(p)})$$

is determined by the pairing $\langle -|-\rangle : K^0K \otimes K_0K \to \mathbb{Z}_{(p)}$, i.e.,

$$\langle \psi^a | f(w) \rangle = f(a).$$

This extends to a continuous pairing given by

$$\langle \psi^a | f \rangle = f(a)$$

if $a \in \mathbb{Z}_p^{\times}$ and $f \in \mathrm{Cont}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p)$; here ψ^a is best viewed as an element of the pro-group ring

$$\mathbb{Z}_p[\![\mathbb{Z}_p^{\times}]\!] \cong (K^0K)_p^{\widehat{}}$$

For more details on $K_0(K)$ and $\mathrm{Cont}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p)$, see [10, section 3]; for a broader overview of the connections with p-adic analysis see [14].

We also recall that K_0K is a bicommutative $\mathbb{Z}_{(p)}$-Hopf algebra with coproduct Ψ given by

$$\Psi(f(w)) = f(w \otimes w)$$

and antipode χ given by

$$\chi(f(w)) = f(w^{-1}).$$

Using the linear pairing $\langle -|-\rangle$ we can obtain a left action of K^0K on K_0K; for $\alpha \in K^0K$, we write $\alpha f(w)$ for this. In particular, if $a \in \mathbb{Z}_{(p)}^{\times}$ this coincides with the action of the Adams operation ψ^a,

$$\psi^a f(w) = f(a^{-1}w).$$

The reason for the inverse is that we are using the standard left action of the dual of the Hopf algebra K_0K defined by

$$\alpha x = \sum_i \langle \alpha(\chi(x_i')) | x_i'' \rangle ,$$

where $\Psi x = \sum_i x_i' \otimes x_i''$, $\Psi(g(w)) = g(w \otimes w)$ and $\chi(g(w)) = g(w^{-1})$.

In the p-complete setting, (stable) Adams operations are indexed by the p-adic units $\mathbb{Z}_p^{\times} \subseteq \mathbb{Z}_p$. It follows that there is a continuous action

$$\mathbb{Z}_p^{\times} \times K_r(X)_p^{\widehat{}} \to K_r(X)_p^{\widehat{}}\,; \quad (\alpha, x) \mapsto \psi^{\alpha}(x).$$

We use notation from [13, chapter IX] and the more recent [11]. For an $\mathcal{E}_\infty$ ring spectrum A there is a natural power operation $\mathrm{Q}\colon K_0^{\vee}A \to K_0^{\vee}A$ (sometimes also called θ) satisfying properties that can be deduced from those listed in [13, theorem IX.3.3] for the homology theories $K_*(-;p^r)$ with coefficients, and are discussed in [11, section 6], although the version there is for $\mathbb{Z}/2$-graded K-theory. However, as we are mainly interested in the case of $K_*^{\vee}K$ which is concentrated in even degrees, we work mostly with $K_0^{\vee}(-)$ but sometimes need to relate this to $K_{2n}^{\vee}(-)$ for an integer n.

The operation Q is neither additive nor multiplicative, but it satisfies the identities

$$\mathrm{Q}(x+y) = \mathrm{Q}\,x + \mathrm{Q}\,y + \frac{1}{p}\Big(x^p + y^p - (x+y)^p\Big), \tag{2.2a}$$

$$\mathrm{Q}(xy) = y^p\,\mathrm{Q}\,x + x^p\,\mathrm{Q}\,y + p\,\mathrm{Q}\,x\,\mathrm{Q}\,y, \tag{2.2b}$$

or equivalently the operation $\widehat{\mathrm{Q}}$ defined by

$$\widehat{\mathrm{Q}}\,x = p\,\mathrm{Q}\,x + x^p$$

is additive and multiplicative,

$$\widehat{\mathrm{Q}}(x+y) = \widehat{\mathrm{Q}}\,x + \widehat{\mathrm{Q}}\,y,$$
$$\widehat{\mathrm{Q}}(xy) = \widehat{\mathrm{Q}}\,x\,\widehat{\mathrm{Q}}\,y.$$

We also have $\mathrm{Q}\,1 = 0$, hence $\widehat{\mathrm{Q}}\,1 = 1$ and $\widehat{\mathrm{Q}}$ is a (unital) ring homomorphism. Finally, if $a \in \mathbb{Z}_{(p)}$ and $u \in \mathbb{Z}_{(p)}^{\times}$,

$$\begin{aligned}\mathrm{Q}(ax) &= a\,\mathrm{Q}(x) + \frac{(a-a^p)}{p}x^p,\\ \widehat{\mathrm{Q}}(ax) &= a\,\widehat{\mathrm{Q}}\,x,\\ \psi^u\,\mathrm{Q}(x) &= \mathrm{Q}(\psi^u x).\end{aligned}$$

When $K_r^{\vee}(A) = K_r(A)^{\widehat{}}_p$, the operations Q and $\widehat{\mathrm{Q}}$ are continuous with respect to the p-adic topology. This allows us to extend these identities to the case where $\alpha \in \mathbb{Z}_p^{\times}$,

$$\begin{aligned}\mathrm{Q}(\alpha x) &= \alpha\,\mathrm{Q}(x) + \frac{(\alpha-\alpha^p)}{p}x^p,\\ \psi^\alpha\,\mathrm{Q}(x) &= \mathrm{Q}(\psi^\alpha x),\\ \widehat{\mathrm{Q}}(\alpha x) &= \alpha\,\widehat{\mathrm{Q}}\,x,\\ \psi^\alpha\,\widehat{\mathrm{Q}}(x) &= \widehat{\mathrm{Q}}(\psi^\alpha x).\end{aligned}$$

Suppose that X is an infinite loop space (and so $\Sigma^\infty_+ X$ is an $\mathcal{E}_\infty$ ring spectrum). If $K_0(\Sigma^\infty_+ X)$ is $\mathbb{Z}_{(p)}$-free so that $K_0^{\vee}(\Sigma^\infty_+ X) = K_0(\Sigma^\infty_+ X)^{\widehat{}}_p$ is pro-free, the diagonal map on X induces a coalgebra structure on $K_0(\Sigma^\infty_+ X)$ and a topological coalgebra structure on $K_0^{\vee}(\Sigma^\infty_+ X)$. In that situation, $\widehat{\mathrm{Q}}$ is a coalgebra morphism; in particular, $\widehat{\mathrm{Q}}$ preserves coalgebra primitives.

We also mention a useful fact about Adams operations. Let $\alpha \in \mathbb{Z}_p^{\times}$ and suppose that $\psi^\alpha x = \alpha^d x$. Since ψ^α is a ring homomorphism,

$$\begin{aligned}\psi^\alpha\,\widehat{\mathrm{Q}}\,x &= p\,\mathrm{Q}(\psi^\alpha x) + (\psi^\alpha x)^p\\ &= p\,\mathrm{Q}(\alpha^d x) + (\alpha^d x)^p\\ &= \widehat{\mathrm{Q}}(\alpha^d x),\end{aligned}$$

giving the identity

$$\psi^\alpha\,\widehat{\mathrm{Q}}\,x = \alpha^d\,\widehat{\mathrm{Q}}\,x.$$

3 Power operations on $K_0^\vee K$ and on $K_0^\vee KO$ for $p = 2$

For the case of $K_0^\vee K$ we continue to assume that p is an arbitrary prime.

We begin with the action of Q on the basic element $w \in K_0K \subseteq K_0^\vee K$. For $a \in \mathbb{Z}_{(p)}^\times$,

$$\psi^a \,\mathrm{Q}(w) = \mathrm{Q}(\psi^a w) = \mathrm{Q}(a^{-1}w).$$

We write $\mathrm{Q}(w) = f_0(w)$ where $f_0 \in \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ is the function given by $x \mapsto f_0(x)$, so we are identifying w with the inclusion function $\mathbb{Z}_p^\times \to \mathbb{Z}_p$ under the isomorphism (2.1).

By [13, theorem IX.3.3(vi)], for $k \in \mathbb{Z}$,

$$\mathrm{Q}(kw) = k\,\mathrm{Q}(w) + \frac{(k - k^p)}{p} w^p,$$

so as $\mathbb{Z}_{(p)}^\times \subseteq \mathbb{Z}_p^\times$ is dense, this defines a continuous function

$$\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p; \quad (x, y) \mapsto x f_0(y) + \frac{(x - x^p)}{p} y^p.$$

Taking $y = 1$, this restricts to the continuous function

$$\mathbb{Z}_p^\times \to \mathbb{Z}_p; \quad x \mapsto x f_0(1) + \frac{(x - x^p)}{p},$$

and as $f_0(1) = 0$, we have

$$f_0(x) = \frac{(x - x^p)}{p}.$$

Hence we have

$$\mathrm{Q}\,w = f_0(w) = \frac{(w - w^p)}{p}. \tag{3.1}$$

For $n \in \mathbb{N}$, by [13, theorem IX.3.3(vii)]

$$\mathrm{Q}(w^{n+1}) = w^p\,\mathrm{Q}(w^n) + w^{np}\,\mathrm{Q}(w) + p\,\mathrm{Q}(w^n)\,\mathrm{Q}(w)$$

and an easy induction gives the general formula

$$\mathrm{Q}(w^n) = \frac{(w^n - w^{np})}{p}$$

for all natural numbers. We also have

$$0 = \mathrm{Q}(1) = \mathrm{Q}(w^n w^{-n}) = w^{np}\,\mathrm{Q}(w^{-n}) + w^{-np}\,\mathrm{Q}(w^n) + p\,\mathrm{Q}(w^n)\,\mathrm{Q}(w^{-n})$$

and so

$$\mathrm{Q}(w^{-n}) = \frac{w^{-n} - w^{-np}}{p}.$$

Therefore for all $n \in \mathbb{Z}$,

$$\mathrm{Q}(w^n) = \frac{w^n - w^{np}}{p}. \tag{3.2}$$

The operation $\widehat{\mathrm{Q}}$ is given by

$$\widehat{\mathrm{Q}}(w^n) = \widehat{\mathrm{Q}}(w)^n,$$

so for any $g \in \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ we have

$$\widehat{\mathrm{Q}}(g(w)) = g(\widehat{\mathrm{Q}}\, w) = g(w),$$

and therefore

$$\mathrm{Q}(g(w)) = \frac{1}{p}(g(w) - g(w)^p).$$

This shows that the sequence of polynomial functions defined recursively by $\theta_0(w) = w$ and for $n \geqslant 1$,

$$\theta_n(w) = \frac{1}{p}(\theta_{n-1}(w) - \theta_{n-1}(w)^p),$$

is also given by

$$\theta_n(w) = \mathrm{Q}(\theta_{n-1}(w)). \tag{3.3}$$

It is known that a (topological) $\mathbb{Z}_p$-basis for $K_0^\vee K$ can be made using monomials in the $\theta_n(w)$, see [4] for example. One interpretation of what we have shown is the following result which seems to have been long known to Mike Hopkins *et al*, but we do not know a published source; a referee has drawn our attention to Mark Behrens' article [15, chapter 12, section 6] which contains a related moduli-theoretic interpretation of such θ-algebras which may lead to similar results. We interpret the operation Q as a realisation of an action of θ and therefore $K_0^\vee K$ becomes a p-complete $\mathbb{Z}_p$-θ-algebra [12, 11].

Proposition 3.1. The p-complete $\mathbb{Z}_p$-θ-algebra $K_0^\vee K$ is generated by the element w. Hence $K_0^\vee K$ is a quotient of the free p-complete $\mathbb{Z}_p$-θ-algebra $K_0^\vee(\mathbb{P}S^0)$, namely

$$K_0^\vee K \cong \mathbb{Z}_p[\theta^s(w) : s \geqslant 0]^\wedge_p \Big/ (\!(\theta^s(w)^p - \theta^s(w) + p\theta^{s+1}(w) : s \geqslant 0)\!).$$

Here the quotient is taken with respect to the p-adic closure of the ideal generated by the stated elements, indicated by the use of $(\!(-)\!)$ rather than $(-)$. This shows that apart from the p-adic completion involved, $K_0^\vee K$ is a colimit of Artin-Schreier extensions of the form

$$\mathbb{Z}_p[X]/(X^p - X + pa)$$

whose mod p reduction is the étale $\mathbb{F}_p$-algebra

$$\mathbb{F}_p[X]/(X^p - X) \cong \prod_{0 \leqslant r \leqslant p-1} \mathbb{F}_p.$$

Our discussion also shows that the antipode of $K_0^\vee(K)$, χ satisfies

$$\chi\,\mathrm{Q} = \mathrm{Q}\,\chi. \tag{3.4}$$

Suppose that A is an $\mathcal{E}_\infty$ ring spectrum (or a $K(1)$-local $\mathcal{E}_\infty$ ring spectrum). Then we may consider $K_\bullet^\vee(A)$ where $K_\bullet^\vee(-)$ denotes the $\mathbb{Z}/2$-graded p-complete theory. The power operation Q intertwines with the coaction as described in [8, (2.5)], giving

$$\Psi\,\mathrm{Q}\,x = \mathrm{Q}(\Psi x) \tag{3.5}$$

since the antipode χ satisfies (3.4) and we have a simpler situation compared to ordinary mod p homology where the dual Steenrod algebra supports two distinct Dyer-Lashof structures related by the antipode.

We now give a brief description of the modification required to describe power operations in $K_0^\vee KO$ at the prime $p = 2$. For $KO_*KO_{(2)}$, results of [2, 3] give

- for all $m \in \mathbb{Z}$, $KO_m KO_{(2)} \cong KO_m \otimes KO_0 KO_{(2)}$;
- $KO_0 KO_{(2)}$ is a countable free $\mathbb{Z}_{(2)}$-module;
- $KO_0 KO_{(2)} = \{f(w) \in \mathbb{Q}[w^2, w^{-2}] : f(\mathbb{Z}_2^\times) \subseteq \mathbb{Z}_2\}$.

Passing to $K_0^\vee KO$, recalling that the squaring homomorphism

$$\mathbb{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbb{Z}_2) \to 1 + 8\mathbb{Z}_2 \subseteq \mathbb{Z}_2^\times$$

is surjective, the natural $\mathcal{E}_\infty$ morphism $KO \to KU$ induces a monomorphism of 2-complete θ-algebras $K_0^\vee(KO) \to K_0^\vee(K)$ coinciding with the inclusion of the continuous functions factoring through $(-)^2$.

It is clear that Q restricts to $K_0^\vee KO$ and is given by

$$\mathrm{Q}(f) = \frac{(f - f^2)}{2}.$$

The following elements defined inductively provide a topological basis for $K_0^\vee KO$:

$$\Theta_0(w) = \frac{1 - w^2}{8}, \qquad \Theta_n(w) = \frac{\Theta_{n-1}(w) - \Theta_{n-1}(w)^2}{2} \quad (n \geqslant 1).$$

Then the distinct monomials $\Theta_0(w)^{\varepsilon_0}\Theta_1(w)^{\varepsilon_1} \cdots \Theta_\ell(w)^{\varepsilon_\ell}$ with $\varepsilon_j = 0, 1$ form a topological basis. Here is the analogue of Proposition 3.1.

Proposition 3.2. The 2-complete $\mathbb{Z}_2$-θ-algebra $K_0^\vee KO$ is a quotient of the free 2-complete $\mathbb{Z}_2$-θ-algebra generated by the element $\Theta_0(w)$, i.e.,

$$K_0^\vee KO \cong \mathbb{Z}_2[\Theta_s(w) : s \geqslant 0]\widehat{_2} / ((\Theta_s(w)^2 - \Theta_s(x) + 2\Theta_{s+1}(x) : s \geqslant 0)).$$

4 The completed K-theory of free algebras

In this section we will describe $K_0^\vee(\mathbb{P}X)$, at least for spectra X for which $K_0^\vee X$ is suitably restricted. For our purposes, it will suffice to assume that X is a CW spectrum with only finitely many even dimensional cells. It will be useful to examine how $K_0^\vee(\mathbb{P}X)$ behaves for such complexes.

Suppose that the $(n-1)$-skeleton $X^{[n-1]}$ of X is defined. Then the n-skeleton $X^{[n]}$ is a pushout defined by a diagram of the form

$$\begin{tikzcd} \bigvee_i S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \bigvee_i D^n \arrow[d] \\ X^{[n-1]} \arrow[r] & X^{[n]} \end{tikzcd}$$

for a finite wedge of spheres $\bigvee_i S^{n-1}$. Similarly there is a pushout diagram of commutative S-algebras

$$\begin{tikzcd} \mathbb{P}(\bigvee_i S^{n-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}(\bigvee_i D^n) \arrow[d] \\ \mathbb{P}(X^{[n-1]}) \arrow[r] & \mathbb{P}(X^{[n]}) \end{tikzcd}$$

so $(\mathbb{P}X)^{\langle n\rangle} = \mathbb{P}(X^{[n]})$ is the $\mathcal{E}_\infty$ n-skeleton of the CW commutative S-algebra $\mathbb{P}X$.

If the cells of X are all even dimensional, we only encounter pushout diagrams of the form

$$\begin{tikzcd} \mathbb{P}(\bigvee_i S^{2m-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}(\bigvee_i D^{2m}) \arrow[d] \\ (\mathbb{P}X)^{\langle 2m-2\rangle} \arrow[r] & (\mathbb{P}X)^{\langle 2m\rangle} \end{tikzcd}$$

where

$$(\mathbb{P}X)^{\langle 2m\rangle} \cong (\mathbb{P}X)^{\langle 2m-2\rangle} \wedge_{\mathbb{P}(\bigvee_i S^{2m-1})} \mathbb{P}(\bigvee_i D^{2m}).$$

To calculate $K^\vee_*((\mathbb{P}X)^{\langle 2m\rangle})$ we may use a Künneth spectral sequence of the form

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{K^\vee_*(\mathbb{P}(\bigvee_i S^{2m-1}))}_{s,t}(K^\vee_*((\mathbb{P}X)^{\langle 2m-2\rangle}), K_*) \implies K^\vee_{s+t}((\mathbb{P}X)^{\langle 2m\rangle}), \tag{4.1}$$

where the internal t grading is in $\mathbb{Z}/2$, i.e., it is an integer modulo 2. This is essentially described in [16], but we will require its multiplicativity, and also the fact that it inherits an action of power operations. The latter structure is constructed in a similar fashion to the mod p Dyer-Lashof operations in [25].

Proposition 4.1. The spectral sequence (4.1) collapses at E^2 to give

$$K^\vee_{s+t}((\mathbb{P}X)^{\langle 2m\rangle}) = K^\vee_{s+t}((\mathbb{P}X)^{\langle 2m-2\rangle})[\mathrm{Q}^s\, x_i : s \geqslant 0,\ i\,]\widehat{_p},$$

where each x_i is in even degree.

Proof. Recall from [11] that

$$K^\vee_*\left(\mathbb{P}\left(\bigvee_i S^{2m-1}\right)\right) = \Lambda(z_i)\widehat{_p},$$

the p-completed exterior algebra on odd degree generators $z_i \in K^\vee_1(\mathbb{P}(\bigvee_i S^{2m-1}))$, each of which originates on a wedge summand.

The E^2-term is a divided power algebra over $K^\vee_*((\mathbb{P}X)^{\langle 2m-2\rangle})$ on generators of bidegree $(1,1)$, each represented in the cobar complex by $[\mathrm{Q}^s\, z_i]$. We will write $\gamma_r([\mathrm{Q}^s\, z_i])$ for the r-th divided power of this element and recall that the particular elements $\gamma_{(r)}([\mathrm{Q}^s\, z_i]) = \gamma_{p^r}([\mathrm{Q}^s\, z_i])$ generate the algebra subject to relations of the form

$$\gamma_{(r)}([\mathrm{Q}^s\, z_i])^p = \binom{p^{r+1}}{p^r, \ldots, p^r}\gamma_{(r+1)}([\mathrm{Q}^s\, z_i]),$$

where the multinomial coefficient satisfies

$$\binom{p^{r+1}}{p^r, \dots, p^r} = pt$$

for some integer t not divisble by p. For degree reasons there can only be trivial differentials, so the only issue still to be resolved is that of the multiplicative structure.

We follow a line of argument similar to that of [25]. In the spectral sequence we have

$$\mathrm{Q}[z_i] = [\mathrm{Q}\, z_i],$$

so it only remains to relate this element to a p-th power in the target of the spectral sequence. By [13, chapter IX, theorem 3.3(viii)], if Z_i is represented by $[z_i]$, then $Z_i^p + p\,\mathrm{Q}\,Z_i$ is represented by $[\mathrm{Q}\, z_i]$, therefore Z_i^p is represented by

$$(1-p)[\widehat{\mathrm{Q}}\, z_i] \equiv [\widehat{\mathrm{Q}}\, z_i] \pmod{p}.$$

It follows that each such Z_i has non-trivial p-th power also represented in the 1-line. By induction this can be extended to show that each $\gamma_{(r)}([\mathrm{Q}^s\, z_i])$ represents an element with non-trivial p-th power. Finally, an easy argument shows that the target is a completed polynomial algebra as stated. Q.E.D.

It is also useful to generalise this to the case of a CW spectrum Y with chosen 0-cell $S^0 \to Y$, where $S^0 \xrightarrow{\sim} S$ is the functorial cofibrant replacement of S in the model category of S-modules. We may then consider the reduced free commutative S-algebras $\widetilde{\mathbb{P}}Y$ which is defined as the homotopy pushout of the diagram of solid arrows

$$\begin{tikzcd}
\mathbb{P}S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}Y \arrow[d, dotted] \\
S \arrow[r, dotted] & \widetilde{\mathbb{P}}Y
\end{tikzcd}$$

where the vertical map is the canonical multiplicative extension of $S^0 \to S$; see [7] for more on this construction. As a particular case, we can consider a map $f\colon S^{2m-1} \to S^0$ and form its mapping cone $C_f = S^0 \cup_f D^{2m}$. Then take $S//f = \widetilde{\mathbb{P}}C_f$ to be a homotopy pushout for the diagram

$$\begin{tikzcd}
\mathbb{P}S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}C_f \arrow[d, dotted] \\
S \arrow[r, dotted] & S//f
\end{tikzcd}$$

and there is an associated Künneth spectral sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{K^\vee_*(\mathbb{P}S^0)}(K_*, K^\vee_*(\mathbb{P}C_f)) \implies K^\vee_{s+t}(S//f). \tag{4.2}$$

It is easily seen that

$$K^\vee_*(\mathbb{P}S^0) = \mathbb{Z}_p[\mathrm{Q}^s\, x_0 : s \geqslant 0]^{\widehat{}}_p$$

is a subalgebra of
$$K_*^\vee(\mathbb{P}C_f) = \mathbb{Z}_p[\mathrm{Q}^s\, x_0, \mathrm{Q}^s\, x_{2m} : s \geqslant 0]^\wedge_p,$$
and the spectral sequence has
$$\mathrm{E}^2_{0,*} = K_* \otimes_{K_*^\vee(\mathbb{P}S^0)} K_*^\vee(\mathbb{P}C_f) = \mathbb{Z}_p[\mathrm{Q}^s\, x_{2m} : s \geqslant 0]^\wedge_p, \qquad \mathrm{E}^2_{r,*} = 0 \quad (r \geqslant 1).$$
This discussion establishes

Proposition 4.2. We have
$$K_*^\vee(S/\!/f) = \mathbb{Z}_p[\mathrm{Q}^s\, x_{2m} : s \geqslant 0]^\wedge_p.$$

Provided we know the coaction for $K_*^\vee(C_f)$, that for $K_*^\vee(S/\!/f)$ follows formally. In general we have only the following possible form of coaction,
$$\Psi(x_{2m}) = w^m \otimes x_{2m} + c(f)(1 - w^m),$$
where $c(f)$ is a certain kind of rational number. Then
$$\Psi(\mathrm{Q}^s\, x_{2m}) = \mathrm{Q}^s(\Psi x_{2m})$$
which involves iterated application of Q.

5 Some examples based on elements of Hopf invariant 1

Throughout this section we assume that $p = 2$.

We will consider the examples $S/\!/\eta$ and $S/\!/\nu$ previously discussed in [9]. Similar considerations apply to other examples constructed using elements in the image of the J-homomorphism at an arbitrary prime. In order to study these examples, it is necessary to determine the $K_0^\vee K$-coaction on $K_0^\vee(S/\!/f)$. Our goal is to explain why the following algebraic results holds.

Theorem 5.1. There are continuous epimorphisms of 2-complete $\mathbb{Z}_2$-θ-algebras
$$K_0^\vee(S/\!/\eta) \to K_0^\vee K, \quad K_0^\vee(S/\!/\nu) \to K_0^\vee K,$$
where in each case the domain is a free θ-algebra. Moreover, these are induced by morphisms of $\mathcal{E}_\infty$ ring spectra $S/\!/\eta \to K$ and $S/\!/\nu \to K$.

Proof. We give the ingredients required for the case of η, the other being similar.

We will use the following elements $\Phi_s = \Phi_s(w)$ $(s \geqslant 0)$ of $K_0^\vee K$:
$$\Phi_0 = \frac{(1-w)}{2}, \qquad \Phi_n = \frac{(\Phi_{n-1} - \Phi_{n-1}^2)}{2} \quad (n \geqslant 1). \tag{5.1}$$
By results of [4], $K_0^\vee K$ has a topological basis consisting of the monomials
$$\Phi_0^{\varepsilon_0}\Phi_1^{\varepsilon_1}\cdots\Phi_\ell^{\varepsilon_\ell} \quad (\varepsilon_i = 0, 1). \tag{5.2}$$
If we view these as continuous functions on $\mathbb{Z}_2^\times$, then for a 2-adic unit α expressed as
$$\alpha = 1 - (2a_0 + 2^2 a_1 + \cdots + 2^{r+1} a_r + \cdots)$$

with $a_r = 0, 1$, in $\mathbb{Z}_2$ we have
$$\Phi_r(\alpha) \equiv a_r \pmod 2.$$
We also know that $\mathrm{Q}\,\Phi_s = \Phi_{s+1}$, hence $\Phi_s = \mathrm{Q}^s\,\Phi_0$.

In the case where $f = \eta$, we can take the generator x_2 to have coaction
$$\Psi(x_2) = \Phi_0 \otimes 1 + w \otimes x_2 = \Phi_0 + wx_2, \tag{5.3}$$
where we suppress the tensor product symbols when the meaning seems clear without them. For the coproduct in $K_0^\vee K$ we have
$$\Psi\Phi_0 = \Phi_0 \otimes 1 + w \otimes \Phi_0,$$
and also
$$\Psi\,\mathrm{Q}\,x_2 = w\,\mathrm{Q}\,x_2 + w\Phi_0 x_2^2 - w\Phi_0 x_2 + \Phi_1.$$
Without further calculation we see that there is a homomorphism of topological comodule algebras
$$\mathbb{Z}_2[x_2]\widehat{_2} \to K_0^\vee K; \quad x_2 \mapsto \Phi_0.$$
This is induced from a morphism of $\mathcal{E}_\infty$ ring spectra $S//\eta \to K$ arising from the fact that the composition of $\eta\colon S^1 \to S$ with the unit $S \to K$ is null homotopic. Therefore there is an extension to a continuous epimorphism
$$K_0^\vee(S//\eta) \to K_0^\vee K; \quad \mathrm{Q}^s\,x_2 \mapsto \Phi_s.$$
This displays $K_0^\vee K$ as a quotient of the free θ-algebra $K_0^\vee(S//\eta)$ as in Proposition 3.1. Q.E.D.

Theorem 5.2. There is a $K(1)$-local equivalence
$$S//\eta \xrightarrow{\sim} \prod_{j\geqslant 0} K.$$

Outline of Proof. We will use the homology theory $K(1)_*(-)$, i.e., mod 2 K-theory. For the spectra we are considering, odd degree groups are trivial so we can consider the ungraded $\mathbb{F}_2$-vector spaces obtained from $K(1)_0(-)$. This functor takes values in the category of $K(1)_0(K)$-comodules, where $K(1)_0(K) \subseteq K(1)_0(K(1))$ is the subHopf algebra called the Morava stabiliser (Hopf) algebra and often denoted (rather confusingly) $K(1)_0K(1)$ in the literature.

Using the basis of (5.2), we see that the group-like element $w = 1 - 2\Theta_0 \in K_0^\vee(K)$ reduces mod 2 to 1 and this is the only group-like element of $K(1)_0(K)$. The reductions mod 2 of this basis give a basis for $K(1)_0(K)$ and the increasing coradical filtration $F_kK(1)_0(K)$ ($k \geqslant 0$) defined by Laures & Schuster [24, section 2] has
$$F_kK(1)_0(K) = \mathbb{F}_2\{1, \Phi_0, \dots, \Phi_{k-1}\}.$$

The epimorphism $K_0^\vee(S/\!/\eta) \to K_0^\vee(K)$ gives rise to a commutative diagram of $K(1)_0(K)$-comodule algebras of the following shape.

$$\begin{tikzcd}
{} & K(1)_0(S/\!/\eta) \arrow[r, two heads] \arrow[d, "\Psi"'] \arrow[ddl, bend right] & K(1)_0(K) \arrow[d, "\Psi"] \\
{} & K(1)_0(K) \otimes K(1)_0(S/\!/\eta) \arrow[r, two heads] \arrow[d, "\mathrm{counit}\otimes\mathrm{Id}"'] & K(1)_0(K) \otimes K(1)_0(K) \arrow[dl, bend left, two heads, "\mathrm{counit}\otimes\mathrm{Id}"] \\
K(1)_0(K) \arrow[r, leftrightarrow, "\cong"] & \mathbb{F}_2 \otimes K(1)_0(K) &
\end{tikzcd}$$

The coaction for $K(1)_0(S/\!/\eta) = \mathbb{F}_2[\mathrm{Q}^s : s \geqslant 0]$ is computable recursively, for example (5.3) gives

$$\Psi(x_2) = \Phi_0 \otimes 1 + 1 \otimes x_2.$$

and

$$\Psi(\mathrm{Q}\, x_2) = \Phi_1 \otimes 1 + \Phi_0 \otimes (x_2 + x_2^2) + 1 \otimes \mathrm{Q}\, x_2.$$

Now we can use an appropriate version of the classic Milnor-Moore Theorem of [27], see for example Laures & Schuster [24, theorem 2.8], to deduce that

$$K(1)_0(S/\!/\eta) \cong K(1)_0(K) \otimes \mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta),$$

where $\mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta) \subseteq K(1)_0(S/\!/\eta)$ is the subalgebra of primitives. To use this, we need to determine filtration

$$F_k K(1)_0(S/\!/\eta) = \Psi^{-1}(F_k K(1)_0(K) \otimes K(1)_0(S/\!/\eta)) \quad (k \geqslant 0)$$

associated with the coradical filtration. By induction we find that

$$F_k K(1)_0(S/\!/\eta) = \mathbb{F}_2[x_2, \ldots, \mathrm{Q}^{k-1} x_2].$$

We need to check the condition that the surjection $K(1)_0(S/\!/\eta) \to K(1)_0(K)$ is a $\star$-isomorphism as in [24, definition 2.6] (note that as we are working with *left* comodules we need to consider graded *right* primitives). Using an induction on k, we find that the k-graded right primitive subspace is $F_k K(1)_0(S/\!/\eta)$ and this maps onto $F_k K(1)_0(K)$ which is the k-graded right primitive subspace of $K(1)_0(K)$.

Dualising and taking care with the inherent linearly compact topologies and completed tensor products involved, we obtain an isomorphism of left topological $K(1)^0(K)$-modules

$$K(1)^0(S/\!/\eta) \cong K(1)^0(K) \widehat{\otimes} (\mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta))^\dagger,$$

where $V^\dagger$ denotes the set of functionals supported on finite dimensional subspaces of the vector space V. Choosing a topological basis $\{b_\alpha : \alpha \in A\}$ for $(\mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta))^\dagger$, we may lift each b_α to an element $\widetilde{b_\alpha} \in K^0(S/\!/\eta)$ since $K(1)_1(S/\!/\eta) = 0$. This gives a map $S/\!/\eta \to \prod_{\alpha \in A} K$ which induces a $K(1)$-isomorphism, hence it is a $K(1)$-local equivalence. In fact A can be taken to be countable, so we might as well index on the natural numbers. Q.E.D.

Notice that there is an $\mathcal{E}_\infty$ morphism $S/\!/\eta \to kU$ which induces a surjection on $\pi_*(-)$ but not on $H_*(-;\mathbb{F}_2)$. Hence kU cannot be a retract of $S/\!/\eta$ 2-locally or after 2-completion. However, multiplication by the Bott map induces a cofibre sequence

$$\Sigma^2 kU \to kU \to H\mathbb{Z}$$

where $KU \wedge H\mathbb{Z}$ is rational. Therefore $\Sigma^2 kU \to kU$ is a $K(1)$-local equivalence, so it induces an isomorphism on $K^\vee(-)$.

Notice that

$$w^2 = (1-2\Phi_0)^2 = 1 - 4(\Phi_0 - \Phi_0^2) = 1 - 8\Phi_1,$$

so

$$1 - w^2 = 8\Phi_1.$$

Similarly,

$$w^4 = 1 - 16(\Phi_1 - \Phi_1^2) + 48\Phi_1^2,$$

and therefore

$$1 - w^4 = 16(\Phi_1 - \Phi_1^2) - 48\Phi_1^2 = 32\Phi_2 - 48\Phi_1^2.$$

Such identities allow us to describe the groups

$$\operatorname{Ext}^{1,2n}_{K_*K}(K_*, K_*) = \operatorname{Pr} K_{2n}K/(\eta_{\mathrm{L}} - \eta_{\mathrm{R}})K_{2n}$$

that detect the 2-primary part of image of the J-homomorphism through the e-invariant. Here Pr denotes the subgroup of primitive elements which satisfy

$$\Psi(x) = 1 \otimes x + x \otimes 1,$$

and $\eta_{\mathrm{L}}, \eta_{\mathrm{R}}$ denote the left and right units respectively. When $n = 1, 2, 4$, these groups are cyclic with the following orders and generators:

- 2, generator represented by $u\Phi_0$;
- 8, generator represented by $u^2\Phi_1$;
- 16, generator represented by $u^4(2\Phi_2 - 3\Phi_1^2)$.

Here we write $u \in K_2$ for the Bott generator. In the first and last cases, a generator of $(\operatorname{im} J)_{2n-1}$ maps to the generator, but in the middle case only the multiples of $2u^2\Phi_1$ are hit; for details see [26, 28].

For $S/\!/\nu$ and $S/\!/\sigma$,

$$K_0^\vee(S/\!/\nu) = \mathbb{Z}_2[\mathrm{Q}^s\, x_4 : s \geqslant 0]^{\wedge}_2, \quad K_0^\vee(S/\!/\sigma) = \mathbb{Z}_2[\mathrm{Q}^s\, x_8 : s \geqslant 0]^{\wedge}_2,$$

we have the coactions

$$\Psi x_4 = w^2 \otimes x_4 + 2\Phi_1, \quad \Psi x_8 = w^4 \otimes x_8 + 2\Phi_2 - 3\Phi_1^2.$$

Finally, we note that there is an $\mathcal{E}_\infty$ morphism $S/\!/\nu \to kO$ inducing an epimorphism on $\pi_*(-)$ which is not an epimorphism on $H_*(-;\mathbb{F}_2)$. The composition $S/\!/\nu \to kO \to KO$ induces a $K(1)$-local splitting whose proof is similar to that of Theorem 5.2.

Theorem 5.3. There is a $K(1)$-local equivalence

$$S//\nu \xrightarrow{\sim} \prod_{j\geqslant 0} \Sigma^{4\rho(j)} KO,$$

for some numerical function ρ taking values in $\{0,1\}$.

Remark 5.4. The case of $S//\sigma$ should also be amenable to a similar analysis, however we have not found convenient way to formalise an argument for this case.

References

[1] J. F. Adams, *Stable Homotopy and Generalised Homology*, Chicago Lectures in Mathematics, University of Chicago Press, 1995 Reprint of the 1974 original

[2] J. F. Adams and F. W. Clarke, *Stable operations on complex K-theory*, Ill. J. Math. **21** (1977), 826–829

[3] J. F. Adams, A. S. Harris, and R. M. Switzer, *Hopf algebras of cooperations for real and complex K-theory*, Proc. London Math. Soc. (3) **23** (1971), 385–408

[4] A. Baker, *p-adic continuous functions on rings of integers*, J. Lond. Math. Soc. **33** (1986), 414–20

[5] ———, *I_n-local Johnson-Wilson spectra and their Hopf algebroids*, Documenta Math. **5** (2000), 351–64

[6] ———, *L-complete Hopf algebroids and their comodules*, Contemp. Math. **504** (2009), 1–22

[7] ———, *Calculating with topological André-Quillen theory, I: Homotopical properties of universal derivations and free commutative S-algebras* (2012), available at `arXiv:1208.1868(v5+)`

[8] ———, *Power operations and coactions in highly commutative homology theories*, Publ. Res. Inst. Math. Sci. **51** (2015), 237–272

[9] ———, *Characteristics for E_∞ ring spectra*, Contemp. Math. **708** (2018), 1–17

[10] A. Baker and B. Richter, *On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory*, Commentarii Math. Helv. **80** (2005), 691–723

[11] T. Barthel and M. Frankland, *Completed power operations for Morava E-theory*, Algebr. Geom. Topol. **15** (2015), 2065–2131

[12] A. K. Bousfield, *On p-adic λ-rings and the K-theory of H-spaces*, Math. Z. **223** (1996), no. 3, 483–519

[13] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lect. Notes in Math., vol. 1176, 1986

[14] F. W. Clarke, *p-adic analysis and operations in K-theory*, Groupe de travail d'analyse ultramétrique **14** (1986–7), no. 15, available at `http://www.numdam.org/item?id=GAU_1986-1987__14__A7_0`

[15] D. L. Douglas, J. Francis, A. G. Henriques, and M. A. Hill, *Topological Modular Forms*, Math. Surv. and Mono., vol. 201, 2014

[16] A. D. Elmendorf, I. Kříž, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Vol. 47, 1997 With an appendix by M. Cole

[17] J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), 438–453

[18] M. J. Hopkins, *$K(1)$-local E_∞-ring spectra*, Math. Surv. and Mono. **201** (2014), 287–302

[19] M. Hovey, *Morava E-theory of filtered colimits*, Trans. Amer. Math. Soc. **360** (2008), no. 1, 369–382

[20] ———, *Some spectral sequences in Morava E-theory*, available at `http://mhovey.web.wesleyan.edu`

[21] M. Hovey and N. P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666

[22] M. Joachim, *Higher coherences for equivariant K-theory*, Lond. Math. Soc. Lect. Note Ser. **315** (2004), 87–114

[23] S. O. Kochman, *Homology of the classical groups over the Dyer-Lashof algbera*, Trans. Amer. Math. Soc. **185** (1973), 83–136

[24] G. Laures and B. Schuster, *Towards a splitting of the $K(2)$-local string bordism spectrum*, Proc. Amer. Math. Soc. **147** (2019), 399–410

[25] H. Ligaard and I. Madsen, *Homology operations in the Eilenberg-Moore spectral sequence*, Math. Z. **143** (1975), 45–54

[26] H. R. Miller, D. C. Ravenel, and W. S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. (2) **106** (1977), no. 3, 469–516

[27] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264

[28] D. C. Ravenel, *A novice's guide to the Adams-Novikov spectral sequence*, Lect. Notes in Math. **658** (1978), 404–475

[29] C. Rezk, *The congruence criterion for power operations in Morava E-theory*, Homology Homotopy Appl. **11** (2009), 327–379

[30] ———, *Modular isogeny complexes*, Algebr. Geom. Topol. **12** (2012), 1373–1403

[31] ———, *Rings of power operations for Morava E-theories are Koszul* (2017), available at `arXiv:1204.4831`